International Journal of Theoretical Physics, Vol. 35, No. 12, 1996

Electromagnetic Interpretation of the Massless Spin-1 Field Equation in Curved Space-Time

Antonio Zecca[1]

Received January 17, 1996

The source-free Maxwell equations associated to the massless spin-1 free field equation are considered in curved space-time. The gauge invariance of the theory is discussed by using as starting point the notion of the spinor potential. The structure of the electromagnetic field in the case of the Robertson–Walker space-time is discussed by using the solutions of the massless spin-1 equations previously determined. The flat-universe case of the standard cosmology is studied exactly and considered in some limiting physical situations.

1. INTRODUCTION

The spinor formulation of the complete source-free Maxwell equations in curved space-time $\nabla_{[a}F_{bc]} = 0$, $\nabla_a F^{ab} = 0$ can be expressed, as is well known, in terms of a symmetric massless spin-1 field ϕ satisfying the equation $\nabla^{A\dot{A}}\phi_{AB} = 0$ (e.g., Penrose and Rindler 1986). The theory is established using as starting point a vector potential from which a tensor is constructed that is interpreted to represent the electromagnetic tensor field and that possesses the usual gauge invariance. The formulation is consistent in a general space-time and it does not suffer from the consistency problem of similar equations for higher spin (Buchdahl, 1958, 1962, 1982; Wünsch, 1978; Penrose and Rindler, 1986; Illge, 1992).

The potential can also be introduced in a general way in tensor form as directly associated to the spinor ϕ, independent of the equations of motion and possessing the gauge invariance of the flat space-time model (Illge, 1988, 1992).

In this paper we reconsider the source-free Maxwell equation in spinor form along the line of the work by Illge. The theory is developed by taking

[1] Dipartimento di Fisica dell' Universita' and INFN, Sezione di Milano, I-20133 Milan, Italy.

into account the standard 1–1 correspondence (given by the σ-matrices) between tensors and spinors. Some remarks on the gauge invariance of the theory are given and it is shown that the theory can always be described by spinor potentials corresponding not only to complex but also to real vector potentials.

The structure of the electromagnetic tensor field is determined in the case of the Robertson–Walker space-time. The calculations are performed by using the separated solutions of the massless spin-1 field equation obtained elsewhere (Zecca, 1996). The results are then applied to the standard cosmological model for r, t near zero. In the flat RW space-time the solutions of the field equations can be given in a complete analytical form. As a consequence, the electromagnetic tensor can be obtained explicitly. In the case of the Einstein–deSitter model the physical situation relative to large r, t values is considered.

2. PRELIMINARY RESULTS

We recall some definitions and preliminary results concerning spinors and tensors in a four-dimensional space-time of class C^∞ with metric signature $(+, -, -, -)$. Since we will deal with massless spin-1 field equations whose formulation is consistent in general, no further properties on the space-time manifold will be required. The standard 1–1 correspondence between complex tensors of rank n and spinors of type (n, n) will be denoted by $\leftrightarrow$. The correspondence is provided by the Infeld–van der Waerden quantities $\sigma^i_{A\dot{X}}$ having, among others, the properties

$$\sigma^i_{A\dot{X}} = \overline{\sigma^i_{X\dot{A}}}, \qquad \sigma_a^{A\dot{X}}\sigma^a_{B\dot{Y}} = \delta^A_B\delta^{\dot{X}}_{\dot{Y}} \tag{1}$$

It is immediate that if $A_l \leftrightarrow \chi_{A\dot{X}}$, then

$$A_l = \overline{A_l} \Leftrightarrow \chi_{A\dot{X}} = \overline{\chi}_{\dot{X}A} \tag{2}$$

Another standard result concerns *bivectors*, that is, real antisymmetric tensors of second rank. A second-order tensor F_{ik} is a bivector if and only if its spinor equivalent has the form (Penrose and Rindler, 1986)

$$F_{ik} \leftrightarrow \phi_{AB}\epsilon_{\dot{X}\dot{Y}} + \epsilon_{AB}\overline{\phi}_{\dot{X}\dot{Y}} \tag{3}$$

where ϕ is a symmetric spinor. Moreover, if A_l is the complex vector field equivalent to the spinor field $\chi_{A\dot{X}}$, then

$$\nabla_{[i}A_{l]} \leftrightarrow \tfrac{1}{2}[\epsilon_{AB}\nabla_{C(\dot{X}}\chi^C_{\dot{Y})} - \epsilon_{\dot{X}\dot{Y}}\nabla^{\dot{Z}}_{(A}\chi_{B)\dot{Z}}] \tag{4}$$

The relation easily follows by decomposing the expression $\nabla_{A\dot{X}}\chi_{B\dot{Y}} - \nabla_{B\dot{Y}}\chi_{A\dot{X}}$ into parts symmetric and antisymmetric in A, B (e.g., Pirani, 1964).

Proposition 1 (Illge, 1992). Let $\phi_{AB} = \phi_{BA}$ be a given symmetric spinor field. Then there exist a divergence-free spinor field χ ($\nabla^{A\dot{X}}\chi_{A\dot{X}} = 0$) such that

$$\phi_{AB} = \nabla^{\dot{X}}_{B}\chi_{A\dot{X}} \tag{5}$$

Conversely, if χ is a spinor field satisfying the equation

$$\phi_{AB} = \nabla^{\dot{X}}_{(B}\chi_{A)\dot{X}} \tag{6}$$

and ω is a scalar function, then the spinor field

$$\hat{\chi}_{A\dot{X}} = \chi_{A\dot{X}} + \nabla_{A\dot{X}}\omega \tag{7}$$

satisfies equation (6), too.

After Proposition 1 it is natural to call the spinor χ satisfying equation (5) the *potential* of the field ϕ. The potential χ is defined up to a gauge transformation generated by the scalar function ω as in equation (7). If also $\hat{\chi}$ is required to be divergence-free, then ω is subject to the condition $\nabla^{l}\nabla_{l}\omega = 0$ as follows from equation (7), χ being divergence-free.

Even if the results of Proposition 1 do not make use of equations of motion, the analogy with the usual relation between the potential and the electromagnetic field is enlightening. Supported by the degrees of freedom of the gauge invariance, this analogy can be made more stringent.

Proposition 2. The spinor potential of Proposition 1 can always be chosen to have both properties

$$\chi_{B\dot{Y}} = \overline{\chi}_{\dot{Y}B}, \qquad \nabla^{A\dot{X}}\chi_{A\dot{X}} = 0 \tag{8}$$

Proof. Let $\chi_{A\dot{X}}$ be a fixed divergent-free solution of equation (5) and let A_l be its vector equivalent. To obtain the result, consider the gauge transformation $\hat{\chi}_{A\dot{X}} = \chi_{A\dot{X}} + \nabla_{A\dot{X}}\omega$ and require $\hat{\chi}_{A\dot{X}} = \overline{\hat{\chi}}_{\dot{X}A}$. This implies $\chi_{A\dot{X}} - \overline{\chi}_{\dot{X}A} = \nabla_{\dot{X}A}\overline{\omega} - \nabla_{A\dot{X}}\omega$ or, in terms of the corresponding equivalent vectors, $A_i - \overline{A}_i = \nabla_i\overline{\omega} - \nabla_i\omega = \partial_i(\overline{\omega} - \omega)$. By setting $b_k = \Im A_k$, $\omega_2 = \Im\omega$, we have that the gauge function is then subject to the constraint

$$b_k = -\partial_k\omega_2 \tag{9}$$

The first condition of the proposition is therefore satisfied by choosing ω_2 to be a solution of equation (9), that is,

$$\omega_2 = -\sum_{i=1}^{4}\int^{x_i} b_i(y)\, dy_i \tag{10}$$

Consider now that

$$\nabla^{A\dot{X}}\hat{\chi}_{A\dot{X}} = \nabla^{A\dot{X}}\nabla_{A\dot{X}}\omega = \nabla^{k}\nabla_{k}\omega_1 + i\nabla^{k}\nabla_{k}\omega_2 \qquad (\omega_1 = \Re\omega)$$

By using also equation (9) we have $\nabla^{k}\nabla_{k}\omega_2 = -\nabla^{k}b_k = 0$ because $\chi_{A\dot{X}} \leftrightarrow A_l$

and ∇^k is a real operator. Therefore $\hat{\chi}_{A\dot{X}}$ is divergence-free, too, if and only if ω_I is a solution of $\nabla^k\nabla_k\omega_I = 0$. ■

Even with a potential having the properties (8), the theory is still invariant under a gauge transformation generated by a gauge function ω subject to $\nabla^k\nabla_k\Re\omega = 0$, with $\Im\omega$ being taken fixed.

Moreover, if the potential χ satisfies the first condition (8), then by taking the complex conjugate of equation (6) we get

$$\bar{\phi}_{\dot{X}\dot{Y}} = \nabla^D_{(\dot{X}}\chi_{\dot{Y})D} \tag{11}$$

Therefore, by combining equations (3), (4), (6), and (11) with $A_l \leftrightarrow \chi_{A\dot{X}}$ we have

$$2\nabla_{[i}A_{k]} + F_{ik} = 0 \tag{12}$$

which represents the connection of the field with the potential in tensor form.

3. THE SOURCE-FREE MAXWELL EQUATIONS

The complete source-free Maxwell equations can be written in spinor form as

$$\nabla^{A\dot{X}}\phi_{AB} = 0 \tag{13}$$

ϕ is a symmetric spinor. These equations are consistent in arbitrary curved space-time (Wünsch, 1978; Penrose and Rindler, 1986). If, according to Proposition 1, χ is the spinor potential of the spinor ϕ,

$$\phi_{AB} = \nabla^{\dot{X}}_{(A}\chi_{B)\dot{X}} \tag{14}$$

then as a consequence of equation (13) and the Ricci identities in spinor form, χ satisfies the equation

$$\nabla^{\dot{Z}}_{B}\nabla^{A}_{(\dot{X}|}\chi_{A|\dot{Z})} = 0 \tag{15}$$

and conversely if χ satisfies equation (15), then ϕ defined by equation (14) satisfies equation (13) (Illge, 1992). If one assumes the Lorentz gauge, that is, $\nabla^{A\dot{X}}\chi_{A\dot{X}} = 0$ for the potential χ (see Propositions 1 and 2), then equation (15) can be developed to give (Illge, 1992)

$$\nabla_{A\dot{Y}}\nabla^{A\dot{Y}}\chi_{B\dot{X}} - 2\Box_{AB}\chi^{A}_{\dot{X}} = 0 \tag{16}$$

and hence, in terms of the curvature spinors Φ, X (e.g., Penrose and Rindler, 1986),

$$\nabla_{A\dot{Y}}\nabla^{A\dot{Y}}\chi_{B\dot{X}} - 2\Phi_{BA\dot{X}\dot{E}}\chi^{A\dot{E}} + 6\Lambda\chi_{B\dot{X}} = 0 \tag{17}$$

($\Lambda = \frac{1}{6}X^{AB}_{AB}$), which is equivalent to

$$\nabla^d \nabla_d A_a - R_{ad} A^d = 0 \tag{18}$$

$A_l \leftrightarrow \chi_{A\dot{X}}$ and R_{ab} is the Ricci tensor. Equation (18) represents the tensor form of the equation for the electromagnetic potential in a curved space-time.

According to the previous considerations, the electromagnetic field can equivalently be described in tensor form by equations (18) and (12) or in spinor form by equations (13) and (17), and the spinor potential χ can always be chosen as corresponding to a real vector potential.

To complete the physical interpretation, we develop the expression

$$F_{ab} = \sigma_a^{A\dot{X}} \sigma_b^{B\dot{Y}} (\phi_{AB} \epsilon_{\dot{X}\dot{Y}} + \epsilon_{AB} \overline{\phi}_{\dot{X}\dot{Y}}) \tag{19}$$

using the generalized Pauli spin matrices

$$\sigma_a^{A\dot{X}} = \frac{1}{2} \begin{pmatrix} n_a & -m_a^* \\ -m_a & l_a \end{pmatrix} \tag{20}$$

$\{l^i, n^i, m^i, m^{*i}\}$ is the given null tetrad frame (e.g., Chandrasekhar, 1983). We get

$$\begin{aligned} F_{ab} = \tfrac{1}{2}\{&\phi_{00}(-n_a m_b^* + n_b m_a^*) + \phi_{11}(-m_a l_b + l_a m_b) \\ &+ \phi_{01}(n_a l_b - m_a^* m_b + m_b^* m_a - l_a n_b) + \text{CC}\} \end{aligned} \tag{21}$$

$a, b = 0, 1, 2, 3$. If one considers the Minkowski tetrad

$$\begin{aligned} l^a &\equiv (1/\sqrt{2})(1, 0, 0, 1) \\ n^a &\equiv (1/\sqrt{2})(1, 0, 0, -1) \\ m^a &\equiv (1/\sqrt{2})(0, 1, -i, 0) \\ m^{*a} &\equiv (1/\sqrt{2})(0, 1, i, 0) \end{aligned} \tag{22}$$

then the σ-matrices (20) become the Pauli matrices modulo $1/\sqrt{2}$ and the corresponding F_{ab} components are interpreted to give by definition the electric and magnetic 3-vector fields **E**, **B** (e.g., Penrose and Rindler, 1986):

$$\begin{aligned} F_{01} &\equiv \tfrac{1}{4}(\phi_{00} - \phi_{11} + \overline{\phi_{00}} - \overline{\phi_{11}}) = E_1 \\ F_{02} &\equiv \tfrac{i}{4}(\phi_{00} + \phi_{11} - \overline{\phi_{00}} - \overline{\phi_{11}}) = E_2 \\ F_{03} &\equiv -\tfrac{1}{2}(\phi_{01} + \overline{\phi_{01}}) = E_3 \\ F_{12} &\equiv \tfrac{i}{2}(\phi_{01} - \overline{\phi_{01}}) = -B_3 \\ F_{13} &\equiv -\tfrac{1}{4}(\phi_{00} + \overline{\phi_{00}} + \phi_{11} + \overline{\phi_{11}}) = B_2 \\ F_{23} &\equiv \tfrac{i}{4}(\phi_{11} - \overline{\phi_{11}} - \phi_{00} + \overline{\phi_{00}}) = -B_1 \end{aligned} \tag{23}$$

4. THE ROBERTSON–WALKER SPACE-TIME

The solution of the theory developed in the previous sections could be done, in principle, by applying to equation (13) the generalized Kirchhoff–d'Adhemar formula (e.g., Penrose and Rindler, 1986). An alternative way is directly in the line of the Newman and Penrose (1962) formalism. By making use of the relation $\nabla_{A\dot{X}} = \sigma^a_{A\dot{X}}\nabla_a$ and of the explicit expression of the covariant derivatives in terms of the tabulated spin coefficients (Chandrasekhar, 1983; Penrose and Rindler, 1986), we find that equation (13) is equivalent to a system of four coupled linear equations each of the first order in the directional derivatives.

In the case of the Robertson–Walker space-time of metric

$$g_{\mu\nu} = \text{diag}\{1, -R^2(t)/(1 - ar^2), -R^2(t)r^2, -R^2(t)r^2 \sin^2\theta\}$$
$$(a = 0, \pm 1)$$

and by using the tetrad

$$\begin{aligned} l_a &\equiv \frac{1}{\sqrt{2}}\left(1, -\frac{R}{\sqrt{1-ar^2}}, 0, 0\right) \\ n_a &\equiv \frac{1}{\sqrt{2}}\left(1, \frac{R}{\sqrt{1-ar^2}}, 0, 0\right) \\ m_a &\equiv \frac{rR}{\sqrt{2}}(0, 0, -1, -i \sin\theta) \\ m_a^* &\equiv \frac{rR}{\sqrt{2}}(0, 0, -1, i \sin\theta) \end{aligned} \tag{24}$$

such a system of equations can be explicitly integrated by a separation method that generalizes the Chandrasekhar–Teukolski procedure (Zecca, 1996). The separated solutions have the form

$$\begin{aligned} \phi_{00} &= \phi_0(r)S_0(\theta)\exp(im\varphi)\,T(t) \\ \phi_{01} &= \phi_{10} = \phi_1(r)S_1(\theta)\exp(im\varphi)\,T(t) \\ \phi_{11} &= \overline{\phi_0}(r)S_2(\theta)\exp(im\varphi)\,T(t), \qquad m = 0, \pm1, \pm2, \ldots \\ T(t) &= \frac{B_0}{R^2(t)}\exp\left[-ik\int_0^t \frac{dt'}{R(t')}\right] \end{aligned} \tag{25}$$

$R(t)$ is determined by the underlying cosmological model; k is an integration constant. The explicit form of the real angular functions S_0, S_1, S_2 for $a = 0, 1, -1$ and of the functions ϕ_1, $\phi_0 = \overline{\phi_2}$ near $r = 0$ for $a = \pm 1$ and in a complete form for $a = 0$ can be found in Zecca (1996). Accordingly the electromagnetic field becomes

$$
\begin{aligned}
F_{tr} &= -\frac{B_0 S_1(\theta)}{R(t)\sqrt{1 - ar^2}}\left\{\cos\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)\Re\phi_1(r)\right.\\
&\left.\quad - \sin\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)\Im\phi_1(r)\right\}\\
F_{t\theta} &= \frac{B_0}{2R(t)}\, r\left\{\cos\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) - S_2(\theta))\Re\Phi_0(r)\right.\\
&\left.\quad - \sin\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) + S_2(\theta))\Im\Phi_0(r)\right\}\\
F_{t\phi} &= \frac{B_0}{2R(t)}\, r\sin\theta\left\{\sin\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) + S_2(\theta))\Re\Phi_0(r)\right.\\
&\left.\quad + \cos\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) - S_2(\theta))\Im\Phi_0(r)\right\} \qquad (26)\\
F_{r\theta} &= \frac{B_0}{2\sqrt{1 - ar^2}}\, r\left\{\cos\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) + S_2(\theta))\Re\Phi_0(r)\right.\\
&\left.\quad - \sin\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) - S_2(\theta))\Im\Phi_0(r)\right\}\\
F_{r\phi} &= \frac{B_0}{2\sqrt{1 - ar^2}}\, r\left\{\sin\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) - S_2(\theta))\Re\Phi_0(r)\right.\\
&\left.\quad + \cos\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)(S_0(\theta) + S_2(\theta))\Im\Phi_0(r)\right\}\\
F_{\theta\phi} &= B_0 \sin\theta\, S_1(\theta) r^2\left\{\sin\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)\Re\phi_1(r)\right.\\
&\left.\quad + \cos\left(m\varphi - k\int_0^t \frac{dt'}{R(t')}\right)\Im\phi_1(r)\right\}
\end{aligned}
$$

The structure (26) now will be considered in special physical situations.

(A) As mentioned above, the behavior of the spinor field near $r = 0$ is known:

$$\phi_d \sim r^{[(4\lambda^2+1)^{1/2}-3]/2} \qquad (d = 0, 1) \qquad (27)$$

$\lambda^2 = l(l+1)$, $l \geq |m|$, $l = 2, 3, \ldots$. If we now assume the cosmological background to be given by the standard matter-dominated cosmology, then (Kolb and Turner, 1990)

$$R(t) = \frac{K_a}{1/2 - |a|/6}\, t^{1/2+|a|/6} \qquad (a = 0, \pm 1) \tag{28}$$

for $t \to 0$, K_a a constant. Therefore for small t, r equations (26)–(28) give

$$\begin{aligned}
F_{tr} &\sim \frac{-C_{01}S_1(\theta)}{t^{1/2+|a|/6}} \cos(m\varphi - K_a k t^{1/2-|a|/6})\, r^{[(4\lambda^2+1)^{1/2}-3]/2} \\
F_{t\theta} &\sim \frac{C_{02}}{t^{1/2+|a|/6}} [S_0(\theta) - S_2(\theta)] \cos(m\varphi - K_a k t^{1/2-|a|/6})\, r^{[(4\lambda^2+1)^{1/2}-1]/2} \\
F_{t\phi} &\sim \frac{C_{03}\sin\theta}{t^{1/2+|a|/6}} [S_0(\theta) + S_2(\theta)] \sin(m\varphi - K_a k t^{1/2-|a|/6})\, r^{[(4\lambda^2+1)^{1/2}-1]/2} \qquad (29) \\
F_{r\theta} &\sim C_{12}[S_0(\theta) + S_2(\theta)] \cos(m\varphi - K_a k t^{1/2-|a|/6})\, r^{[(4\lambda^2+1)^{1/2}-1]/2} \\
F_{r\phi} &\sim C_{13}[S_0(\theta) - S_2(\theta)] \sin(m\varphi - K_a k t^{1/2-|a|/6})\, r^{[(4\lambda^2+1)^{1/2}-1]/2} \\
F_{\theta\phi} &\sim C_{23}\sin\theta\, S_1(\theta) \sin(m\varphi - K_a k t^{1/2-|a|/6})\, r^{[(4\lambda^2+1)^{1/2}+1]/2}, \qquad a = 0, \pm 1
\end{aligned}$$

(B) The flat-universe case of the Robertson–Walker model can be treated exactly. For $a = 0$ we have (Zecca, 1996)

$$\phi_d = e^{-ikr}\phi\left(\frac{(4\lambda^2+1)^{1/2}+1}{2} + d;\ 1 + (4\lambda^2+1)^{1/2};\ 2ikr\right) r^{[(4\lambda^2+1)^{1/2}-3]/2} \tag{30}$$

$d = 0, 1$, $\phi(a; c; x)$ is the confluent hypergeometric function, and k is an integration constant.

It is of some interest to consider the electromagnetic field for large r. By using the asymptotic behavior

$$\phi(a; b; x) \sim \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} + \frac{\Gamma(c)}{\Gamma(c-a)} (-x)^{-a}$$

(Abramovitz and Stegun, 1970), we have from a straightforward calculation that for large r, ϕ_d has the form

$$\begin{aligned}
\phi_d &\sim A_d(\lambda, k) r^{-2+d} \exp\{i[kr + \alpha_d(\lambda)]\} \\
A_d &= \frac{\Gamma(1 + (4\lambda^2+1)^{1/2})}{\Gamma((4\lambda^2+1)^{1/2}/2 + 1/2 + d)} (2k)^{d-[(4\lambda^2+1)^{1/2}+1]/2} \qquad (31) \\
\alpha_d &= \frac{\pi}{2}\left[d - \frac{1}{2} - \frac{1}{2}(4\lambda^2+1)^{1/2}\right], \qquad d = 0, 1
\end{aligned}$$

Suppose now that $R(t)$ is the one determined by the standard cosmological model with comparable contributions to the energy density from both matter and radiation ($a = 0$). Then for large t (Kolb and Turner, 1990)

$$R(t) = \frac{3}{K_0} t^{2/3}, \qquad K_0 \text{ constant} \tag{32}$$

In the limit of both large t and r the electromagnetic tensor has therefore the behavior

$$\begin{aligned}
F_{tr} &\sim \frac{-A_1 B_0 K_0}{3rt^{2/3}} \cos(m\varphi - kK_0 t^{1/3} + kr + \alpha_0)\, S_1(\theta) \\
F_{t\theta} &\sim \frac{A_0 B_0 K_0}{6rt^{2/3}} \{S_0 \cos(m\varphi - kK_0 t^{1/3} + kr + \alpha_0) \\
&\quad - S_2 \cos(m\varphi - kK_0 t^{1/3} - kr - \alpha_0)\} \\
F_{t\phi} &\sim \frac{A_0 B_0 K_0 \sin\theta}{6rt^{2/3}} \{S_0 \sin(m\varphi - kK_0 t^{1/3} + kr + \alpha_0) \\
&\quad + S_2 \sin(m\varphi - kK_0 t^{1/3} - kr - \alpha_0)\} \\
F_{r\theta} &\sim \frac{A_0 B_0}{2r} \{S_0 \cos(m\varphi - kK_0 t^{1/3} + kr + \alpha_0) \\
&\quad + S_2 \cos(m\varphi - kK_0 t^{1/3} - kr - \alpha_0)\} \\
F_{r\phi} &\sim \frac{A_0 B_0}{2r} \{S_0 \sin(m\varphi - kK_0 t^{1/3} + kr + \alpha_0) \\
&\quad - S_2 \sin(m\varphi - kK_0 t^{1/3} - kr - \alpha_0)\} \\
F_{\theta\phi} &\sim A_1 B_0 \sin\theta\, S_1(\theta) r \sin(m\varphi - kK_0 t^{1/3} + kr + \alpha_0)
\end{aligned} \tag{33}$$

The only term that in the considered limits does not approximate zero is $F_{\theta\phi}$, which indeed increases with r. This divergence is not a contradiction on physical grounds because our considerations are based on "monochromatic" solutions [see equations (25), (31)].

Finally we remark that the problem of determining the potential χ in equation (14) once ϕ is a known given solution of (13) does not seem to be easy. Also in the Robertson–Walker model, by developing the right-hand side of equation (14) as in Zecca (1996) one is left with a system of four coupled equations each of the first order in the directional derivatives. These equations do not seem to be separable, even by assuming the known term ϕ to have a separated form like that of equation (25).

REFERENCES

Abramovitz, W., and Stegun, I. E. (1970). *Handbook of Mathematical Functions*, Dover, New York.

Buchdahl, H. A. (1958). *Nuovo Cimento,* **10**, 96.
Buchdahl, H. A. (1962). *Nuovo Cimento,* **25**, 486.
Buchdahl, H. A. (1982). *Journal of Physics A,* **15**, 1.
Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes,* Oxford University Press, Oxford.
Kolb, E. W., and Turner, M. S. (1990). *The Early Universe,* Addison-Wesley, Reading, Massachusetts.
Illge, R. (1988). *General Relativity and Gravitation,* **20**, 551.
Illge, R. (1992). *Zeitschrift für Analysis und ihre Anwendungen,* **11**, 25.
Newman, E. T., and Penrose, R. (1962). *Journal of Mathematical Physics,* **3**, 566.
Penrose, R., and Rindler, W. (1986). *Spinors and Space-Time,* Vols. I and II, Cambridge University Press, Cambridge.
Pirani, F. A. E. (1964). Lectures in general relativity, in *Brandeis Summer Institute in Theoretical Physics,* Vol. I, Prentice-Hall, Englewood Cliffs, New Jersey.
Wünsch, V. (1978). *Beiträge zur Analysis,* **12**, 47.
Zecca, A. (1996). *International Journal of Theoretical Physics,* **35**, 383.